Higher order correlations in quantum chaotic spectra

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The statistical properties of the quantum chaotic spectra have been studied, so far, only up to the second order correlation effects. Numerical as well as analytical evidence that the random matrix theory can successfully model the spectral fluctuatations of these systems is available only up to this order. For a complete understanding of spectral properties it is highly desirable to study the higher order spectral correlations. This will also inform us about the limitations of random matrix theory in modeling the properties of quantum chaotic systems. Our main purpose in this paper is to carry out this study by a semiclassical calculation for the quantum maps; however, results are also valid for time-independent systems. [S1063-651X(97)15402-2]

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I. INTRODUCTION

In generic Hamiltonian systems with many degrees of freedom, the classical dynamics shows an enormous richness in structure, increasing with the interaction between degrees of freedom. The classical motion is mainly of two types, integrable and chaotic; see [1] for details. This paper deals with the quantum properties of Hamiltonians whose classical limit is chaotic.

The strongly chaotic nature of underlying classical dynamics suggests that we intuitively expect some kind of random behavior in quantum dynamics as well. This is because the classical dynamics is indeed a limit ($\hbar=0$) of quantum dynamics, and therefore the nature of former should be somehow reflected in the latter. In fact, various analytical and numerical studies (see [2] and references therein) have confirmed that the manifestation of chaotic behavior in quantum dynamics occurs through randomization (partial or full) of matrices of associated quantum operators. The spectral and strength fluctuations of these operators can be well modeled (up to second order correlations) by one of the various universality classes of random matrices. Most common among these are the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE) and the circular orthogonal ensemble and the circular unitary ensemble [3]. The former pertain to autonomous systems whereas the latter have application in the study of nonautonomous systems such as quantum maps.

The presence of random matrix theory (RMT)-type spectra in quantum chaotic systems can be explained by the Gutzwiller-semiclassical quantization scheme [4] for time-independent systems which uses the elegant technique of path integral sum given by Feynman, and relates the chaotic manifolds of classical dynamics to the eigenfunctions of quantum dynamics. A similar formulation is also given for time-evolution operators of quantum maps [5]. The spectral fluctuation measures can then be determined approximately by using the principle of uniformity [6], which is based on the uniform distribution of periodic orbits at large time scales, and gives a technique to evaluate the sum of periodic orbit contributions. Using this technique for autonomous Hamiltonians, Berry [7] provided an explicit expression for the semiclassical form factor $K_2(\tau)$—the Fourier transform of the two-level spectral correlation function—for values of τ in the range $\tau\ll 1$ (the time measured in units of $2\pi\hbar\overline{d}$, where $\overline{d}$ is the mean spectral density). This result has an exact analogy with the corresponding RMT behavior; following essentially the same technique as used by Berry for autonomous Hamiltonians, this analogy can also be proved for quantum maps [2]. In the region $\tau\gg 1$ also, the limiting behavior was analyzed by Berry using a semiclassical sum rule which makes use of the properties of the function related to the quantum-mechanical density of states.

Notwithstanding the good agreement between RMT and statistical quantum chaos up to second order correlations (long and very long), still there is no reason to believe that the random matrix theory (RMT) can model all nth order spectral as well as strength correlations. The numerical studies for many systems (e.g, Baker map [8], quantum kicked rotor [2], etc.) have already indicated that even second order correlation effects, when considered on short time scales (i.e., very long range correlations), do not follow the random matrix prediction and are nonuniversal. This is the range where the classical dynamics is still diffusive, and periodic orbits are not yet uniformly distributed. The deviation from RMT in this range agrees well with our intution, as one should expect RMT to be applicable only on those time scales where the variables associated with classical dynamics are random enough to fully randomize the matrices associated with corresponding quantum operators. Moreover, the sum rules for the matrix elements of quantum chaotic operators [9] have already been found, differing from those of RMT. But a study of higher order correlations between zeros of Reimann ζ function shows a good agreement with RMT [10,11].

Thus it is relevant to know what properties and up to what order the behavior of quantum operators can be modeled by RMT, and when it ultimately breaks down. Our attempt, in this paper, is to make a comparative study of one such prop-

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erty, namely, the nth order spectral correlation, as all n-level spectral fluctuation measures can be expressed in its terms. We fufill this goal by carrying out a semiclassical study of the Fourier transform of the n-level correlation function R_n; the reason to consider the Fourier tranform lies in the covenience of its analytical as well as numerical calculability. We proceed as follows.

The Gutzwiller formulation gives us the density of states as a sum over periodic orbits, and this gives rise to periodic orbit interaction terms in n-level density correlation function. Berry, in order to obtain result for a two-level form factor, neglected the contribution from these interacting terms as a first order approximation (the so-called diagonal approximation). However, for a complete evaluation of the form factor, one has to calculate the contribution due to interacting terms. The lack of the knowledge of action correlations handicaps us from doing so. One attempt in this direction was made in Ref. [12], in which, by assuming the complete validity of random matrix theory, the periodic orbit correlations were calculated from the RMT form factor. These, when compared with numerically obtained correlations (for Baker map, hyperbola billiard, and perturbed Schrödinger cat map), showed a good agreement. The numerical study of these actions also indicated the presence of an uncorrelated component (exponentially larger than correlated part); in this paper we use this fact. We assume that, on long time scales as a first order approximation, actions are uncorrelated, and we calculate the nth order form factor. Under this approximation, the result turns out to be same as that of RMT, which is also confirmed from the numerical analysis for at least two higher order fluctuation measures given in this paper. For a complete calculation of the nth order form factor, the action correlations which will determine the higher order terms should also be taken into account.

In this paper, we present our semiclassical study for quantum maps, but the method can easily be generalized for time-independent systems, and one obtains the same results. Similarly the RMT is given only for circular ensembles (CE) but, once again, the final results are also valid for Gaussian ensembles (GE), which follows due to GE-CE equivalence for large dimensions).

This paper is organized as follows: In Sec. II A, we briefly review the definition of various random matrix ensembles. For later use, we also discuss the relation between the n-level form factor and correlation functions. Section II B deals with a brief review of the fundamentals of quantum maps and the earlier obtained results for the two-level form factor. Both the Secs. II A and II B are included in this paper so as to clarify the ideas used in Sec. III, which deals with the higher order correlations and form factors for the quantum spectra with exact symmetry. In Sec. IV, we numerically study the higher order fluctuation measures, namely, skewness and excess for a prototype quantum chaotic system (that is, kicked rotor), and compare them with those of RMT. We summarize our results in Sec. V.

II. PRELIMINARIES

A. Random matrix results

Here we briefly outline the results for the nth order form factor (i.e, Fourier transform of n-point density correlation function) for eigenvalues of the equilibrium circular ensembles of the random matrix theory.

1. Circular ensembles

The circular-type equilibrium ensembles are ensembles of unitary matrices U_β in which the distinct nonzero matrix elements of U are distributed independently as zero-centered random variables; β defines the number of independent components of the matrix elements of U. There are three such ensembles, characterized by β, namely COE, CUE, and CSE for $\beta=1$, 2, and 4, respectively. These universality classes are determined by the invariance of the system under time-reversal (TR) transformation (or more generally antiunitary transformation) and are described by the invariance of the ensemble measure: invariance under orthogonal or symplectic transformations for TR-invariant systems and under unitary transformations for TR-noninvariant ones. The invariance restricts the allowed space of matrices, for example, to that of symmetric unitary matrices for orthogonal invariance.

2. n-point correlators

The aforementioned unitarity of U implies that its eigenvalues $\exp(iE_j)$ lie on the unit circle in the complex plane, where exponents E_j's are termed as eigenangles. The density of states is then defined by

$$\rho(E)=\sum_{j=1}^{N}\sum_{k=-\infty}^{\infty}\delta(E-2\pi k-E_j) \tag{2.1}$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp(inE)\mathrm{Tr}(U^n) \tag{2.2}$$

and has the mean value $\langle\rho\rangle=N/2\pi$.

For analytical studies of the spectrum, it is the usual practice to calculate the level density correlations. For cases where the level density $\rho(E)$ can be written as the sum of a smooth part $\langle\rho(E)\rangle$ and a fluctuating component $\delta\rho(E)$, it is preferable to study the correlations R_k between the fluctuating parts of the density. The R_k's can be defined as follows:

$$R_k(E_1,\ldots,E_k)=\frac{\langle\delta\rho(E_1)\delta\rho(E_2)\cdots\delta\rho(E_k)\rangle}{\langle\rho(E_1)\rangle\cdots\langle\rho(E_k)\rangle}. \tag{2.3}$$

Here $\delta\rho(E)=\rho(E)-\langle\rho(E)\rangle$, where $E_j=E+l_jD$ for $j=1,2,\ldots,k-1$ and $E_k=E$, with D as the mean spacing and $\langle\,\rangle$ implying the averaging over variable E for ranges containing sufficient number of mean spacings. By subtracting $\langle\rho(E)\rangle$ from Eq. (2.1) and using notation $\mathrm{Tr}(U^n)=t_n$, $\delta\rho(E)$ can further be written as follows:

$$\delta\rho(E)=\frac{1}{2\pi}\sum_{n=-\infty,\neq 0}^{\infty}t_n\exp(inE). \tag{2.4}$$

The substitution of Eq. (2.4) into Eq. (2.3) gives

$$R_k(E_1,\ldots,\quad E_k)=\frac{1}{N^k}\sum_J \langle t_{j_1}t_{j_2}\cdots t_{j_k}\rangle \times\left\langle \exp\left[i\left(\sum_{m=1}^{k} j_m\right)E\right]\right\rangle_E \times\exp\left[iD\left(\sum_{m=1}^{k-1} j_m l_m\right)\right]. \tag{2.5}$$

Here Σ_J implies the summation over all indices $j_1,j_2,\ldots,j_k$, with each index varying from $-\infty$ to ∞ except zero (that is, none of the indices take value zero). The averaging over E reduces Eq. (2.5) in the following form:

$$R_k(E_1,\ldots,E_k)=\frac{1}{N^k}\sum_J \langle t_{j_1}t_{j_2}\cdots t_{j_k}\rangle\,\delta\left(\sum_{m=1}^{k} j_m\right)\times\exp\left[iD\left(\sum_{m=1}^{k-1} j_m l_m\right)\right] \tag{2.6}$$

$$=\frac{1}{N^k}\sum_J{}' \langle t_{j_1}t_{j_2}\cdots t_{-(\Sigma_{m=1}^{k-1}j_m)}\rangle \times\exp\left[iD\left(\sum_{m=1}^{k-1} j_m l_m\right)\right]. \tag{2.7}$$

Here Σ' implies Σ_J subjected to the condition that $\Sigma_{m=1}^{k-1}j_m \neq 0$. In the semiclassical analysis, instead of dealing directly with R_k, it is easier to calculate the kth order form factor, defined as follows:

$$K_k(\tau_1,\ldots,\tau_{k-1})=\int \exp\left[2\pi i\sum_{j=1}^{k-1}(E_j-E_k)\tau_j\right]\times R_k(E_1,\ldots,E_k)dE_1\cdots dE_{k-1} \tag{2.8}$$

$$=\int \exp\left[2\pi i\sum_{j=1}^{k-1} l_j\tau_j\right]\times R_k(l_1,\ldots,l_{k-1})dl_1\cdots dl_{k-1}. \tag{2.9}$$

Substitution of Eq. (2.5) into Eq. (2.7) gives K_k in terms of the traces,

$$K_k=\frac{1}{N^k}\sum_J{}' \langle t_{j_1}t_{j_2}\cdots t_{-(\Sigma_{m=1}^{k-1}j_m)}\rangle \prod_{m=1}^{k-1}\delta\left(\tau_m+\frac{j_m}{N}\right). \tag{2.10}$$

As in this study we confine ourselves to a calculation of K_k only for $|\tau_m|<1$, $m=1,2,\ldots(k-1)$, for these values of $|\tau_m|$'s only those terms in Σ_J contribute which have indices $\{j_1,j_2,\ldots,j_k\}$ much less than N. Therefore, Σ_J' in Eq. (2.5) can be replaced by Σ_G', in which the indices vary from some value $-n$ to n, where $n<N$ with all other conditions the same.

The above result for K_k can further be simplified by using the recently obtained result for the statistics of the traces [13], which indicates that the first few traces $t_1,t_2,\ldots,t_n$ of large unitary matrices (N large) taken from any of the circular ensembles display no noticeble correlation. The ensemble average of each of the traces vanishes (that is, $\langle t_n\rangle=0$) for all the circular ensembles, due to uniformity of the distribution of E_i's, the eigenangles; see Ref. [13] for details. But note that t_n and t_{-n} are not independent of each other, and one can show that [3,13]

$$\langle|t_n|^2\rangle=\beta n/N,\quad n\ll N, \tag{2.11}$$

where β is 1 or 2 depending on whether the ensemble is COE or CUE, respectively. Therefore, in general, $\langle\Pi_{j=1}^{n}t_j\rangle=0$, if at least one t_j is such that its opposite $t_{-|j|}$ is not present in the product. This product exists only if the following condition is satisfied:

$$\left\langle\prod_j t_j\right\rangle=\left\langle\prod_l |t_l|^2\right\rangle=\prod_l \langle|t_l|^2\rangle. \tag{2.12}$$

Now, as can be seen from Eq. (2.10), for k odd, every product appearing in the sum contains an odd number of t_j's, and therefore the above condition can never be satisfied. This gives, for k odd,

$$K_{k\text{-odd}}(\tau_1,\tau_2,\ldots,\tau_{k-1})\simeq 0,\quad |\tau_i|_{i=1,\ldots,k}<1. \tag{2.13}$$

On the other hand, the application of condition (2.10) gives the following result for k even:

$$K_{k\text{-even}}(\tau_1,\tau_2,\ldots,\tau_{k-1})\simeq\alpha\sum_P[\delta(\tau_{p_1}+\tau_{p_2})\delta(\tau_{p_3}+\tau_{p_4})\cdots\delta(\tau_{p_{k-3}}+\tau_{p_{k-2}})]\times|\tau_{p_1}||\tau_{p_3}||\tau_{p_5}|\cdots|\tau_{p_{k-1}}|, \tag{2.14}$$

where $\alpha=1$ for CUE and 2 for COE. The Σ_P implies the sum over all possible permutations of indices $p_1,p_2,\ldots,p_{k-1}$ over the set $1,2,\ldots,k-1$.

The result given by Eqs. (2.13) and (2.14) are valid only when each $|\tau|<1$. For cases with $|\tau|\simeq 1$ or >1 (i.e., $n\simeq N$), one has to take into account the correlation between traces. Furthermore, though the method adopted here for the derivation of the K_k result is applicable only for ensembles of unitary matrices, the final results are also valid for Gaussian ensembles. This follows due to the equivalence of fluctuation measures of the circular and Gaussian ensemble in the large dimensionality limit.

B. Quantum map vs classical map

A classical map can be described by a canonical mapping M of the coordinate variable q and momenta variable p at a discrete time step t_n to those at t_{n+1}:

$$\begin{pmatrix} q_{n+1}\\ p_{n+1}\end{pmatrix}=M\begin{pmatrix} q_n\\ p_n\end{pmatrix}, \tag{2.15}$$

with $W(q_{n+1},q_n)$ as the generator of the map such that

$$p_n=-\frac{\partial W(q_{n+1},q_n)}{\partial q_n},\quad p_{n+1}=\frac{\partial W(q_{n+1},q_n)}{\partial q_{n+1}}. \tag{2.16}$$

The nature of the time step considered can give rise to different kind of maps [1]. For example, for time-periodic

Hamiltonians, it is easier to study the dynamics in terms of fixed time steps (i.e., the period of the Hamiltonian), the related mapping known as stroboscopic mapping. For time-independent systems it is sometimes sufficient to consider only those steps of dynamics which occur on a definite plane; that is, intersections of a trajectory with a plane (instead of equal time steps), known as Poincare mapping.

The quantization of a two-dimensional classical map, when the phase space upon which it acts is compact, leads to the construction of unitary matrices U of a finite dimension N, and their semiclassical limit is obtained for $N\to\infty$. For example, for a canonical mapping on a two-dimensional torus (here taken to be a two-dimensional phase space with periodicities Q and P in q and p directions, respectively), the corresponding quantum propagator acts in an N-dimensional Hilbert space and is represented by an $N\times N$ unitary matrix U. This follows because the number of states N allowed to be associated, by quantization, with the finite classical space is restricted (uncertainty principle); N is determined by the following relation:

$$2\pi\hbar N=QP. \tag{2.17}$$

Here N plays the role of the inverse of Planck's constant, with $N\to\infty$ as semiclassical limit.

C. Semiclassical form factor for quantum maps: Symmetry preserving cases

For quantum maps acting in a finite Hilbert space, Eqs. (2.3) and (2.9) can be used to write the quantum-mechanical two-level form factor $K_2(\tau)$,

$$K_2(\tau)=\frac{2\pi}{N^2}\int_0^1 dr\left\langle \delta\rho\left(E+\frac{r\pi}{N}\right)\delta\rho\left(E-\frac{r\pi}{N}\right)\times\exp(2\pi i r\tau)\right\rangle_E. \tag{2.18}$$

Using Eq. (2.4) in Eq. (2.18), $K_2(k)$ can further be reduced in the following form:

$$K_2(\tau)=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\exp[ik(E_n-E_m)]-N\delta_{n,0} \tag{2.19}$$

$$=\frac{1}{N}|\mathrm{Tr}(U^n)|^2-N\delta_{n,0}, \tag{2.20}$$

where $n=N\tau$. Now the semiclassical expression of $K_2(\tau)$ can be obtained from above equation by using semiclassical form of $\mathrm{Tr}(U^n)$ which can be expressed as a sum over periodic orbits in classical phase space $(r=(q,p))$ [5],

$$\mathrm{Tr}(U^n)=g\sum_j\sum_{m_j}A_j^{\{m_j\}}\exp\left[i\frac{m_jW_j}{\hbar}-i\pi\nu_j/2\right]. \tag{2.21}$$

Here the amplitude A_j $[=n_j|\partial(r_n-r_0)/\partial r_0|^{-1/2}_{r_0=r_n}=n_j(\sinh\alpha_j)^{-1}]$ of the contribution from each (multiply traversed) periodic orbit j, with period n, depends on the stability α_j of the orbit; for long periodic orbits A_j can be approximated as $A_j\simeq n_j\exp(-\alpha_j)=n_j\exp(-\gamma n_j)$, with γ as the entropy of the classical motion. W_j is the action for one traversal of the orbit, m_j is the number of traversals, $n_j=n/m_j$ is the period of the orbit with a single forward traversal, and ν_j is the Maslov index. The index g refers to the number of symmetric analogs, existing in phase space, for the periodic orbit. While considering the long-range correlations which are mainly affected by the long periodic orbits, it is sufficient to consider $|m_j|=1$ (and therefore $n_j=n$); this follows due to the principle of uniformity which states that on large time scales periodic orbits tend to distribute uniformly in phase space, their density increasing exponentially while the intensity decreases. Thus, on large time scales, long periodic orbits which are almost all primitive dominate the phase space.

The evaluation of $|\mathrm{Tr}(U^n)|^2$, in the semiclassical limit $N\to\infty$, can be done as follows. As is obvious from Eq. (2.21), $|\mathrm{Tr}(U^n)|^2$ contains terms of the type $\exp[i(W_j-W_k)/\hbar]$, and, therefore, for a complete evaluation of $K_2(\tau)$, it becomes important to study the distribution of amplitudes A_j and actions W_j. But, in the semiclassical limit $\hbar\to 0$, the significant contributions comes only from those orbit interactions for which $W_j-W_i\leqslant o(\hbar)$. The contributions from other orbit interactions become negligible, in this limit, due to the presence of the rapid oscillations leading to destructive interferences. Thus, for the leading order semiclassical asymptotics of $|\mathrm{Tr}(U^n)|^2$, one needs to consider only ''diagonal'' terms [7] with $W_i\simeq W_j$, which, in large-n limit (such that $n/N=\tau\ll 1$) can be evaluated by invoking Hannay's sum rule for the amplitudes [6],

$$|\mathrm{Tr}(U^n)|^2\simeq g^2\sum_j A_j^2. \tag{2.22}$$

Now by using Hannay's sum rule for the intensities, which comes from the principle of uniformity [5] and is given by

$$\sum_i A_i^2\delta\left(|\tau|-\frac{n_i}{N}\right)=\frac{N^2|\tau|}{g},\quad |\tau|\ll 1, \tag{2.23}$$

one can obtain the two level form factor $K_2(\tau)$ [7,2], which turns out to be same as that for random matrix ensembles, the under small-τ approximation [Eq. (2.14)]

$$K_2(\tau)\simeq g|\tau|. \tag{2.24}$$

Note that the above result is valid only for $|\tau|\ll 1$, i.e., $|n|\ll N$. This limit of validity comes into existence due to considerations of only diagonal terms in the evaluation of $|\mathrm{Tr}(U^n)|^2$. For cases $|\tau|\simeq 1$, one needs to consider contributions due to the constructive interference of very long periodic orbits [with $W_i-W_j\simeq o(\hbar)$] too, which once again requires an understanding of the distribution of periodic orbit actions. Moreover, in the derivation of spectral densty in terms of periodic orbits, the quasienergy is assumed to be complex, with a very small imaginary part ϵ (required to avoid the divergence of the formula, occurring for real energies). Due to the finiteness of ϵ, the periodic orbits with period $n>n^*$ (that is, the oscillations with energies $\delta E<\epsilon$, where $\delta E=\hbar/n$ and $\epsilon=\hbar/n^*$) cannot be taken into account in this formulation.

The result obtained in Eq. (2.12) for the semiclassical form factor is same as that for exact symmetry classes of RMT, under the same limit. For example, $g=2$ and 1 give corresponding COE and CUE results, respectively [2].

III. HIGHER ORDER SPECTRAL CORRELATIONS

In this paper, we restrain ourselves to the study of the k-level correlation functions in short time limits (where period n of the longest periodic orbit, in phase space, is much greater than unity but much less than N, i.e., $1 \ll n \ll N$ or $|\tau| = n/N \ll 1$), which will allow us to use the principle of uniformity in the evaluation of cross multiplication of k periodic orbit contributions, thus simplifying the calculations. For simplicity, and to explain our method, we first calculate the third and fourth order correlations, and then generalize them to kth order correlations.

A. Third order correlation

For simplicity, let us first calculate the third order correlation function. The substitution of Eqs. (2.4) and (2.21) into Eq. (2.3), with $k=3$, gives us

$$R_3(E_1,E_2,E_3) = \frac{g^3}{N^3}\left\langle \sum_{ijk} \sum_{m_i,m_j,m_k=\pm 1} A_i A_j A_k \times \left\langle \left\{ \exp\left[i m_i \left(n_i E + n_i \ell_1 \frac{2\pi}{N} + \frac{W_i}{\hbar} \right) \right] \times \exp\left[i m_j \left(n_j E + n_j \ell_2 \frac{2\pi}{N} + \frac{W_j}{\hbar} \right) \right] \times \exp\left[i m_k \left(n_k E + \frac{W_k}{\hbar} \right) \right] \right\} \right\rangle \tag{3.1}$$

Here $\langle \rangle$ implies a local averaging with respect to E; that is, the energy averaging over ranges which are classically small but quantum mechanically large, so that a large number of levels are included. For example, a good choice is to take the size of the averaging range to be E itself, i.e., to define

$$\langle f(E) \rangle_E = \frac{1}{E} \int_0^E f(E') dE'. \tag{3.2}$$

Therefore the variation of amplitude, in the above equation, with respect to energy is very small (amplitude being a classical quantity) and can be ignored. This gives

$$R_3(E_1,E_2,E_3) = \frac{g^3}{N^3} \sum_{ijk} \sum_{m_i,m_j,m_k=\pm 1} A_i A_j A_k \times \exp\left[(m_i n_i \ell_1 + m_j n_j \ell_2) \frac{2\pi i}{N} \right] \times \langle \exp[(m_i n_i + m_j n_j + m_k n_k) i E] \rangle \times \left\langle \exp\left[(m_i W_i + m_j W_j + m_k W_k) \frac{i}{\hbar} \right] \right\rangle . \tag{3.3}$$

Due to averaging over E, the contribution of various terms in Eq. (3.3) will be determined by the fact of whether their exponents contain E or not; the terms containing a factor of type $\exp[iEn]$ will not make any contribution. Thus we can divide all the terms into following two classes.

Case (1). Terms with all n_i, n_j, and n_k of the same sign (i.e., either all positive or all negative). On averaging over E, the contribution of these terms to R_3 turns out to be zero due to presence of a factor of type $\exp[\pm i(|n_i| + |n_j| + |n_k|)E]$.

Case (2). Terms with any two among (n_i, n_j, n_k) with the same sign $(+$ or $-)$, and a third one with opposite sign. The terms under this case contain a factor $\exp[\pm(n_i + n_j - n_k)iE]$ (and its permutations). As mentioned above, these terms will make a nonzero contribution if $n_k = n_i + n_j$ (or $n_i = n_j + n_k$, $n_j = n_i + n_k$). Thus Eq. (3.3) can be reduced to the form

$$R_3(E_1,E_2,E_3) = \frac{g^3}{N^3} \sum_{ijk} A_i A_j A_k \sum_{m=\pm 1} \exp\left(\frac{2\pi m i}{N} (n_i \ell_1 + n_j \ell_2) \right) \left\langle \exp\left(\frac{im}{\hbar} (W_i + W_j - W_k) \right) \right\rangle + \exp\left(\frac{2\pi m i}{N} (n_i \ell_1 - n_j \ell_2) \right) \times \left\langle \exp\left(\frac{im}{\hbar} (W_i - W_j + W_k) \right) \right\rangle + \exp\left(\frac{2\pi m i}{N} (n_j \ell_2 - n_i \ell_1) \right) \left\langle \exp\left(\frac{im}{\hbar} (W_j - W_i + W_k) \right) \right\rangle \tag{3.4}$$

with second and third terms corresponding to $n_i = n_j + n_k$ and $n_j = n_i + n_k$, respectively.

To evaluate terms of type $\langle e^{i(W_i + W_j - W_k)/\hbar} \rangle$, we proceed as follows. Here W_i, the action of a periodic orbit with period n_i, can also be written as a sum of n_i single step actions $W_i = \sum_{l=0}^{n_i - 1} W_l(q_{l+1}, q_l)|_{q_{n_i} = q_0}$. For strongly chaotic dynamics and on large tine scales, these single step actions can be regarded as independent variables with a pair-correlation coefficient decaying exponentially to zero. An extension of central limit theorem therefore implies that W_i's are Gaussian random variable on large time scales. Hence $\theta = W_i + W_j - W_k$ will also be a Gaussian random variable with mean zero, the variance (referred to as var θ) of which is given as follows:

$$\mathrm{var}\,\theta = \langle (W_i + W_j - W_k)^2 \rangle \tag{3.5}$$

$$\simeq \langle W_i^2 \rangle + \langle W_j^2 \rangle + \langle W_k^2 \rangle \simeq 3T. \tag{3.6}$$

Here T is the average time period of periodic orbits given by $T = \hbar/\delta E$, where δE is the energy range over which average is taken.

As on large time scales, the phase space is densely and uniforemly covered by periodic orbits, and a typical trajectory can be approximated by a very long periodic orbit. This permits us to approximate the average of $\exp(iW_j)$ over all periodic orbits by a phase-space average. This gives $\langle \exp(i\theta)\rangle = \exp(-\mathrm{var}\theta) = \exp(-3T)$. Here, in Eq. (3.6), the correlations between various actions i.e., terms of type $\langle W_i W_j\rangle$ has been approximated to zero (W_i is assumed to be random variable). But, as mentioned in Sec. I, the correlation between actions is not entirely zero, that is, W_i's are not exactly random variables. The distribution function $P(\theta)$ of these actions can be written as $P(\theta) = P_{\mathrm{random}} + P_{\mathrm{correlated}}$, where the random part of the distribution P_{random} dominates the nonrandom part $P_{\mathrm{correlated}}$. Therefore, on large time scales $P(\theta)$ can be approximated by a Gaussian, which gives us the first order term of R_3. To calculate higher order terms which are not negligible on very long time scales, the correlations between actions must also be taken into account.

To further simplify the calculation of R_3, $A_k(\simeq n_k e^{-\gamma n_k})$, in Eq. (3.4), can be replaced by $A_i A_j(n_i^{-1} + n_j^{-1})$ for the terms which survive due to $n_k = n_i + n_j$. Similarly for terms with $n_i = n_j + n_k$ or $(n_j = n_i + n_k)$, A_k can be replaced by $A_i A_j^{-1} n_j (1 - n_j n_i^{-1})$ and $A_j A_i^{-1} n_i (1 - n_i n_j^{-1})$, respectively. This leads us to following form of R_3:

$$R_3 = \frac{g^3}{N^3} \sum_{ij} A_i^2 A_j^2 \left(\frac{1}{n_i} + \frac{1}{n_j}\right) \sum_{m=\pm 1} \exp\left(\frac{2\pi m i}{N}(n_i \ell_1 + n_j \ell_2)\right)$$
$$+ \sum_{ij} A_i^2 n_j (1 - n_j n_i^{-1}) \times \sum_{m=\pm 1} \exp\left(\frac{2\pi m i}{N}(n_i \ell_1 - n_j \ell_2)\right)$$
$$+ \sum_{ij} A_j^2 n_i (1 - n_i n_j^{-1}) \times \sum_{m=\pm 1} \exp\left(\frac{2\pi m i}{N}(-n_i \ell_1 + n_j \ell_2)\right) e^{-3T}. \tag{3.7}$$

Here the second term corresponds to $n_i = n_j + n_k$ or $n_k = n_i - n_j$, and the third term corresponds to $n_j = n_i + n_k$ or $n_k = n_j - n_i$.

The Fourier transform of R_3 gives us the third order form factor K_3,

$$K_3(\tau_1, \tau_2) = \int e^{2\pi i[(r_1 - r_3)\tau_1 + (r_2 - r_3)\tau_2]} \times R_3(r_1, r_2, r_3) dr_1 dr_2 dr_3 \tag{3.8}$$

$$= \int e^{2\pi i[\ell_1 \tau_1 + \ell_2 \tau_2]} R_3(\ell_1, \ell_2) d\ell_1 d\ell_2 \tag{3.9}$$

(where $r_1 - r_3 = \ell_1$ and $r_2 - r_3 = \ell_2$). Equation (3.9) follows from Eq. (3.8), as R_3 depends only on differences $r_1 - r_3$ and $r_2 - r_3$.

Further calculation of K_3 can be done by substituting Eq. (3.7) in Eq. (3.9), and by making use of following equalities (see the Appendix) which follow from the principle of uniformity:

$$\sum_j n_j^a \delta\left(|\tau_2| - \frac{n_j}{N}\right) = \frac{\langle n^a\rangle}{g} = \frac{f(0,a)}{g} \tag{3.10}$$

and

$$\sum_i \frac{A_i^2}{n_i} \delta\left(|\tau_1| - \frac{n_i}{N}\right) = \frac{N}{g}. \tag{3.11}$$

The result obtained depends on whether τ's are greater or less than zero. This gives rise to the following three possibilities. *Case (1) Both* $\tau_1, \tau_2 > 0$ *or* $\tau_1, \tau_2 < 0$,

$$K_3 = \frac{g^3}{N^3}\left[\sum_{ij} \left(A_i^2 \frac{A_j^2}{n_j} + A_j^2 \frac{A_i^2}{n_i}\right) \delta\left(|\tau_1| - \frac{n_i}{N}\right) \times \delta\left(|\tau_2| - \frac{n_j}{N}\right)\right] e^{-3T} \tag{3.12}$$

$$\simeq g(|\tau_1| + |\tau_2|) e^{-3T}. \tag{3.13}$$

It is obvious from the above equation that K_3 falls very rapidly to zero for large-T values, that is, for time scales on which sufficiently long periodic orbits exist in the phase space. On long time scales, therefore this is similar to the RMT result [Eq. (2.13)]. *Case (2)* $\tau_1 > 0$, $\tau_2 < 0$ *or* $\tau_1 < 0$, $\tau_2 > 0$,

$$K_3 = \frac{g^3}{N^3} \sum_{ij} (A_j^2 n_i + A_i^2 n_j - A_j^2 n_i^2 n_j^{-1} - A_i^2 n_j^2 n_i^{-1}) \times \delta\left(|\tau_1| - \frac{n_i}{N}\right) \delta\left(|\tau_2| - \frac{n_j}{N}\right) e^{-3T} \tag{3.14}$$

$$\simeq g[|\tau_1| f_1(|\tau_2|) + |\tau_2| f_1(|\tau_1|) - f_2(|\tau_1|) - f_2(|\tau_2|)] e^{-3T}. \tag{3.15}$$

On substituting values of $f_1 = f(0,1)$ and $f_2 = f(0,2)$ (see the Appendix) in the above equation, we obtain $K_3 \simeq g(|\tau_1| - |\tau_2|)(e^{N\tau_2} - e^{N\tau_1}) e^{-3T}$, where $N|\tau_1|$ and $N|\tau_2|$ are of the same order as that of T. This results in a nearly zero K_3 on large time scales which is again similar to the RMT result.

Note that the above-mentioned similarity between K_3 results for quantum maps and RMT has been shown here only for those time scales at which principle of uniformity is well applicable to the distribution of periodic orbits. No conclusion can be drawn about the short time scales from the above analysis, although the deviation of two-point fluctuation measures for quantum maps from those of RMT [2,7] suggests that we expect the same for higher orders too.

B. Fourth order correlation

To calculate the fourth order correlation function, we substitute Eqs. (2.4) and (2.21) into Eq. (2.3), with $k = 4$. This gives us

$$R_4(E_1,E_2,E_3,E_4)$$

$$= \frac{g^4}{N^4}\sum_{ijk}\sum_{m_i,m_j,m_k=\pm 1} A_iA_jA_kA_r \times\langle \exp[(m_in_i+m_jn_j+m_kn_k+m_rn_r)iE]\rangle \times \exp\left[(m_in_i\ell_1+m_jn_j\ell_2+m_kn_k\ell_3)\frac{2\pi i}{N}\right] \times\left\langle \exp\left[(m_iW_i+m_jW_j+m_kW_k+m_rW_r)\frac{i}{\hbar}\right]\right\rangle. \tag{3.16}$$

Again the significant contributions to R_4 come from following four types of terms. *Case (1) Terms where pairwise cancellation occurs, i.e., terms with $n_i=n_k, n_j=n_r$ and $W_i=W_k, W_j=W_r$ (and their permutations).* The contribution R_{4i} from such terms can be written as follows:

$$R_{4i}=\frac{g^4}{N^4}\sum_{\text{perm}}\sum_{ij} A_i^2A_j^2 \times \sum_{m=\pm 1}\exp\left(\frac{2\pi mi}{N}(n_i(\ell_1-\ell_3)+n_j\ell_2)\right). \tag{3.17}$$

Here Σ_{perm} refers to the sum over all possible permutations of pairs.

Case (2) Terms with $n_i-n_j-n_k-n_r=0$ (and other such permutations). The contributions to R_4 from terms with $n_i-n_j-n_k-n_r=0$ can be written as follows:

$$\frac{g^4}{N^4}\sum_{ijk} A_i^2\left(n_jn_k-\frac{n_j^2n_k}{n_i}-\frac{n_jn_k^2}{n_i}\right) \times\sum_{m=\pm 1}\exp\left(\frac{2\pi mi}{N}(n_i\ell_1-n_j\ell_2-n_k\ell_3)\right) \times\left\langle \exp\left(\frac{im}{\hbar}(W_i-W_j-W_k-W_r)\right)\right\rangle. \tag{3.18}$$

Similarly one can write contributions from terms with $n_j-n_i-n_k-n_r=0$ and $n_k-n_i-n_j-n_r=0$). The symbol R_{4ii} will refer to the sum of contibutions of all such terms.

The contribution R_{4iii} from a term with $n_r-n_i-n_j-n_k=0$ is

$$R_{4iii}=\frac{g^4}{N^4}\sum_{ij} A_i^2A_j^2A_k^2\left(\frac{1}{n_in_j}+\frac{1}{n_jn_k}-\frac{1}{n_in_k}\right) \times\sum_{m=\pm 1}\exp\left(\frac{2\pi mi}{N}(n_i\ell_1+n_j\ell_2+n_k\ell_3)\right) \times\left\langle \exp\left(-\frac{im}{\hbar}(W_r-W_i-W_j-W_k)\right)\right\rangle. \tag{3.19}$$

Case (3) Terms with $n_i+n_j-n_k-n_r=0$ (and other such permutations). The contribution to R_4 from terms with $n_i+n_j-n_k-n_r=0$ can be written as follows:

$$\frac{g^4}{N^4}\sum_{ijk} A_i^2A_j^2n_k\left(\frac{1}{n_j}+\frac{1}{n_i}-\frac{n_k}{n_in_j}\right) \times\sum_{m=\pm 1}\exp\left(\frac{2\pi mi}{N}(n_i\ell_1+n_j\ell_2-n_k\ell_3)\right) \times\left\langle \exp\left(-\frac{im}{\hbar}(W_i+W_j-W_k-W_r)\right)\right\rangle. \tag{3.20}$$

Similarly one can write the contributions from terms $n_i-n_i-n_k+n_r=0$ and $n_i-n_j+n_k-n_r=0$. The symbol R_{4iv} refers to the sum of contibutions of all such terms.

Thus R_4 can be written as follows:

$$R_4=R_{4i}+R_{4ii}+R_{4iii}+R_{4iv} \tag{3.21}$$

where in each of the contributions R_{4i}–R_{4iv} the terms of type $\langle e^{W_i+W_j+W_k-W_r}\rangle$ can be replaced by e^{-4T} (as done earlier for K_3). The Fourier transform of R_4 gives us the fourth order form factor K_4,

$$K_4(\tau_1,\tau_2,\tau_3)=(2\pi)^3\int e^{2\pi i[\ell_1\tau_1+\ell_2\tau_2+\ell_3\tau_3]} \times R_4(\ell_1,\ell_2,\ell_3)d\ell_1d\ell_2d\ell_3 \tag{3.22}$$

$$=K_i+K_{ii}+K_{iii}+K_{iv}, \tag{3.23}$$

where

$$K_{i,ii,iii,iv}=\int e^{2\pi i[\ell_1\tau_1+\ell_2\tau_2+\ell_3\tau_3]}R_{4(i,ii,iii,iv)} \times(\ell_1,\ell_2,\ell_3)d\ell_1d\ell_2d\ell_3.$$

.

Now K_4 can be calculated by substituting Eqs. (3.17)–(3.20) into Eq. (3.23), and using equalities (3.10) and (3.11). Again, as for K_3, the result depends on whether τ's are greater or less than zero. This gives rise to the following three possibilities.

Case (1) $\tau_1,\tau_2,\tau_3>0$ *or* $\tau_1,\tau_2,\tau_3<0$. In this case, except for K_{4iii}, the contributions from all others, namely, K_{4i}, K_{4ii}, and K_{4iv}, are zero. Thus

$$K_4=K_{4iii}=\frac{g^4}{N^4}\sum_{ijk}\sum_{m=\pm 1} A_i^2A_j^2A_k^2\left(\frac{1}{n_jn_k}+\frac{1}{n_jn_k}+\frac{1}{n_jn_k}\right) \times\delta\left(\tau_1-\frac{mn_i}{N}\right)\delta\left(\tau_2-\frac{mn_j}{N}\right)\delta\left(\tau_3-\frac{mn_k}{N}\right)e^{-4T} \tag{3.24}$$

$$\simeq g(|\tau_1|+|\tau_2|+|\tau_3|)e^{-4T}. \tag{3.25}$$

Due to the presence of the exponentially decaying factor, K_4 turns out to be approximately zero for large T values, which is again similar to RMT results.

Case (2) Any two of τ_1,τ_2,τ_3 positive (negative) and the third one negative (positive). Let $\tau_i,\tau_j>0$ and $\tau_k<0$ (where i, j, and k can take any of the values 1, 2, or 3). In this case,

$$K_{4i}\simeq g[\delta(\tau_i+\tau_k)|\tau_i||\tau_j|+\delta(\tau_j+\tau_k)|\tau_i||\tau_j|], \tag{3.26}$$

$$K_{4ii}=g[|\tau_k|f_1(|\tau_i|)f_1(|\tau_j|)-f_1(|\tau_j|)f_2(|\tau_i|)-f_1(|\tau_i|)f_2(|\tau_j|)]e^{-4T}, \tag{3.27}$$

$$K_{4iii}=0, \tag{3.28}$$

$$K_{4iv}=g[|\tau_i|f_1(|\tau_k|)+|\tau_j|f_1(|\tau_k|)-f_2(|\tau_k|)]e^{-4T}. \tag{3.29}$$

As is obvious from above equations, only K_{4i} does not contain an exponentially decaying factor, and therefore makes a nonzero contribution to K_4 in large time limits. The above results are also valid if $\tau_i,\tau_j<0$ and $\tau_k>0$. A comparison with RMT results shows that this lowest order contribution to K_4 is same as that in RMT for $|\tau_1|,|\tau_2|,|\tau_3|\ll 1$.

C. *k*th order correlation

The method used in calculation of third and fourth order form factors can further be generalized to the kth order correlation function The substitution of Eqs. (2.4) and (2.21) into Eq. (2.3) gives us

$$R_k=\frac{g^k}{N^k}\sum_{i_1,\ldots,i_k}A_{i_1}A_{i_2}\cdots A_{i_k}\sum_{m_1,\ldots,m_k=\pm1}\left\langle \exp\left[im_1\left(n_{i_1}E+\frac{2\pi}{N}n_{i_1}\ell_1+\frac{W_{i_1}}{\hbar}\right)\right]\cdots\exp\left[im_k\left(n_{i_k}E+\frac{W_{i_k}}{\hbar}\right)\right]\right\rangle. \tag{3.30}$$

It can further be rearranged as follows:

$$R_k=\frac{g^k}{N^k}\sum_{i_1,\ldots,i_k}\prod_{l=1}^{k}A_{i_l}\times\sum_{m_1,\ldots,m_k=\pm1}\left\langle\exp\left[iE\sum_{l=1}^{k}m_ln_{i_l}\right]\right\rangle\times\exp\left[\frac{2i\pi}{N}\sum_{l=1}^{k-1}m_ln_{i_l}\ell_l\right]\left\langle\exp\left[\frac{i}{\hbar}\sum_{l=1}^{k}m_lW_{i_l}\right]\right\rangle. \tag{3.31}$$

Due to ±1 values taken by each m_l, $l=1,2,\ldots,k$, there can be 2^k different combinations of n_{i_l}'s in the first exponent of eq.(3.31). Let $M(r)$ be the set of a particular choice of values for each m_l in the set $\{m_1,m_2,\ldots,m_k\}$. Therefore there can exist 2^k such sets, denoted by $M(r)$ with $r=1\rightarrow2^k$, of which only 2^{k-1} sets are distinct. Here two sets $M(r)$ and $M(r')$ are considered indistinct if the values of each m_l in $M(r)$ is oppsite (in sign) to that in $M(r')$. Now Eq. (3.31) can be rewritten as

$$R_k=\frac{g^k}{N^k}\sum_{r=1}^{2^k}F_{M_r}, \tag{3.32}$$

where F_{M_r} is given as follows:

$$F_{M_r}=\sum_{i_1,\ldots,i_k}\prod_{l=1}^{k}A_{i_l}\left\langle\exp\left[iE\sum_{l=1}^{k}m_ln_{i_l}\right]\right\rangle\times\exp\left[i\frac{2\pi}{N}\sum_{l=1}^{k-1}m_ln_{i_l}\ell_l\right]\left\langle\exp\left[\frac{i}{\hbar}\sum_{l=1}^{k}m_lW_{i_l}\right]\right\rangle. \tag{3.33}$$

In the above equation, the values taken by m_l's are the same as for $M(r)$.

Now for the same reason as given for the third order correlation function, only those terms of Eq. (3.33) for which the multiplying factor of E in the exponent is zero will contribute significantly to R_k. Due to k summations over periodic orbits, each summation containing a large number of them, there are many possibilities, resulting in a zero coefficient of E. These various possibilities may arise, due to the ''groupwise cancellation of periods'' in the first exponent of Eq. (3.32), containing various groups of periods in the exponent, where in each group the positive traversals of a few orbits are canceled by the negative traversals of a few other orbits. Let G stand for any division of indices $1,2,\ldots,k$ into q subgroups $(G_1,G_2,\ldots,G_q)$; then a term appearing in Eq. (3.33) will make a nonzero contribution to F_{M_r} if it satisfies following condition:

$$\sum_{G_j}m_ln_{i_l}=0 \quad (j=1,2,\ldots,q), \tag{3.34}$$

where the summation is over indices l present in subgroup G_j, and so on. As is obvious, one of these subgroups will contain index i_k. Later we will need to distinguish it from other subgroups; let us call it G_j'. Thus F_{M_r} can be rewritten as

$$F_{M_r}=\sum_G\left\langle\exp\left[\frac{i}{\hbar}\sum_{l=1}^{p}m_lW_{j_l}\right]\right\rangle\prod_{j=1}^{q}C_{G_j}. \tag{3.35}$$

Here Σ_G implies the summation over all possible divisions of indices $i_1,i_2,\ldots,i_k$ into various subgroups, and Π_j implies the product of the contributions C_{G_j} from all G_j's for one such division, where

$$C_{G_j}=\sum_{j_1,\ldots,j_k}\left(\prod_{l=1}^{p}A_{j_l}\right)\delta\left(\sum_{l=1}^{p}m_ln_{j_l}\right)\exp\left[\frac{2i\pi}{N}\sum_{l=1}^{p}m_ln_{j_l}b_l\right]. \tag{3.36}$$

Here the p indices contained in subgroup G_j are denoted by $j_1,j_2,\ldots,j_p$, with m_l as the coefficient of n_{j_l}, and b_l's refer to ℓ's associated with these indices (e.g., if $j_l=i_1$, then $b_l=\ell_1$, and so on); if $j_l=i_k$, then $b_l=b_k=0$. Note that, in group G_j', the index j_1 is always chosen to be i_k (so as to simplify the presentation) with $b_1=b_k=0$. Now by using $n_{j_1}=\Sigma_{l=2}^{p}\alpha_l n_{j_l}$, with $\alpha_l=-m_l/m_1$, and $A_{j_l}\simeq n_{j_l}e^{-\gamma n_{j_l}}$, one can show that

$$\prod_{l=1}^{p} A_{j_l}\simeq\sum_{l=2}^{p}\alpha_l\frac{A_{j_l}^{\alpha_l+1}}{n_{j_l}^{\alpha_l-1}}\prod_{s=2\neq l}^{p}\frac{A_{j_s}^{\alpha_s+1}}{n_{j_s}^{\alpha_s}}. \tag{3.37}$$

By using Eq. (3.37), C_{G_j} can further be reduced as follows:

$$C_{G_j}=\sum_{j_1,\ldots,j_l}\left(\sum_{l=2}^{p}\alpha_l\frac{A_{j_l}^{\alpha_l+1}}{n_{j_l}^{\alpha_l-1}}\prod_{s=2\neq l}^{p}\frac{A_{j_s}^{\alpha_s+1}}{n_{j_s}^{\alpha_s}}\right)\times\exp\left[\frac{2i\pi}{N}\left(\sum_{l=2}^{p}(b_l-b_1)m_l n_{j_l}\right)\right]. \tag{3.38}$$

Note here that, for $j=j'$, $b_1=0$. It can be seen from the above equations that the most significant contribution to R_k comes from those terms where a pairwise cancellation of time periods $n_i,n_j,n_k,n_s,\ldots$ (appearing as a factor of E in the exponent) as well as actions $W_i,W_j,W_k,W_s,\ldots$ (so that there is no exponential decay) occurs which is possible only when k is even. The contribution R_{ki} from such terms can be written as follows:

$$R_{ki}=\frac{g^k}{N^k}\sum_{\text{perm}}\sum_{j_1,\ldots,j_k}A_{j_1}^2A_{j_3}^2\cdots A_{j_{k-1}}^2\sum_{m=\pm1}\times\exp\left[\frac{2\pi mi}{N}\left(\sum_{l=1}^{(k-2)/2}n_{j_{2l-1}}(b_{2l-1}-b_{2l})+n_{j_{k-1}}b_{k-1}\right)\right]. \tag{3.39}$$

Σ_{perm} implies the summation over all possible permutations of indices $j_1,j_2,\ldots,j_k$ taken from set $i_1,i_2,\ldots,i_k$. The contributions from all other terms contain an exponential term, with a sum over actions (in units of $\hbar$), as its exponent [i.e., term $\langle\exp(\Sigma m_{i_j}W_{i_j})\rangle$]. In large time limits, the application of the central limit theorem (CLT) again permits us to replace this term by e^{-kT}. The contribution of such terms to R_k can be given as follows:

$$R_{kii}=\frac{g^k}{N^k}\sum_r\sum_G^{M_r}\prod_j\sum_{j_1,\ldots,j_p}^{G_j}\left(\sum_{l=2}^{p}\alpha_l\frac{A_{j_l}^{\alpha_l+1}}{n_{j_l}^{\alpha_l-1}}\prod_{s=2\neq l}^{p}\frac{A_{j_s}^{\alpha_s+1}}{n_{j_s}^{\alpha_s}}\right)\times\exp\left[\frac{2i\pi}{N}\left(\sum_{l=2}^{p}(b_l-b_1)m_l n_{j_l}\right)\right]e^{-kT}. \tag{3.40}$$

The substitution of Eqs. (3.39) and (3.40) into Eq. (2.9) gives us following result for K_k:

$$K_k=K_{ki}+K_{kii}, \tag{3.41}$$

where

$$K_{ki}=\frac{g^k}{N^k}\sum_{\text{perm}}\sum_{j_1,\ldots,j_k}A_{j_1}^2A_{j_3}^2\cdots A_{j_{k-1}}\times\sum_{m=\pm1}\delta\left(t_{2k-1}-\frac{mn_{j_{2k-1}}}{N}\right)\times\prod_{l=1}^{(k-1)/2}\delta\left(t_{2l-1}-\frac{mn_{j_{2l-1}}}{N}\right)\delta(t_{2l-1}+\tau_{2l}) \tag{3.42}$$

and

$$K_{kii}=\frac{g^k}{N^k}\sum_r\sum_G^{M_r}\prod_j\sum_{j_1,\ldots,j_p}^{G_j}\left(\sum_{l=2}^{p}\alpha_l\frac{A_{j_l}^{\alpha_l+1}}{n_{j_l}^{\alpha_l-1}}\prod_{s=2\neq l}^{p}\frac{A_{j_s}^{\alpha_s+1}}{n_{j_s}^{\alpha_s}}\right)\times\left[\prod_{l=2}^{p}\delta\left(t_l+\frac{m_l n_{j_l}}{N}\right)\right]\delta\left(\sum_{l}^{G_j}t_l\right)e^{-k\tau}, \tag{3.43}$$

where t_l is the τ variable associated with b_l with $t_l=0$ if $b_l=\ell_k$. By using Eq. (A3), one can further reduce above equations in the following form:

$$K_{ki}=\sum_P[\delta(\tau_{p_1}+\tau_{p_2})\delta(\tau_{p_3}+\tau_{p_4})\cdots(k/2-1)\text{terms}]\times|\tau_{p_1}||\tau_{p_3}|\cdots|\tau_{p_{(k-1}}|\delta_{k,\text{even}}, \tag{3.44}$$

where Σ_P refers to sum over all possible permutations of indices $\{p_1,p_2,\ldots,p_4\}$ taken from set $\{1,2,\ldots,k-1\}$, and

$$K_{kii}=g\sum_r\sum_G^{M(r)}\prod_j D_{G_j}e^{-kT} \tag{3.45}$$

with D_{G_j} given as follows:

$$D_{G_j}=\begin{cases}\delta\left(\sum_{l=1}^{p}t_l\right)\left[\prod_{l=2}^{p}\delta(h_l+m_l)\right]\sum_{l=2}^{p}\alpha_l f(\alpha_l+1,\alpha_l-1)\prod_{s=2,\neq l}f(\alpha_s+1,\alpha_s), & G_j\not\equiv G_j' \quad (3.46)\\ \left[\prod_{l=2}^{p}\delta(h_l-m_l)\right]\sum_{l=2}^{p}\alpha_l f(\alpha_l+1,\alpha_l-1)\prod_{s=2,\neq l}f(\alpha_s+1,\alpha_s), & G_j\equiv G_j', \quad (3.47)\end{cases}$$

with $h_l = \mathrm{sgn}(t_l)$. Here, as obvious, the contribution K_{ki} exist only for for all even order form factors. For odd order form factors, only K_{kii} contributes. But, on large time scales,the presence of an exponential decaying factor makes this contribution very small. Thus, on large time scales (i.e., $n \to \infty$, $N \to \infty$, $n/N < 1$) one can write

$$K_{k-\mathrm{odd}} \simeq 0. \tag{3.48}$$

A comparison of Eqs. (3.44) and (3.47) with Eqs. (2.14) and (2.13) informs us that the results obtained for both odd as well as even order form factors agree well with that of RMT.

IV. NUMERICAL VERIFICATION

This section contains a numerical study of third and fourth order fluctuation measures, namely, skewness γ_1 and excess γ_2. We choose the kicked rotor system for this purpose, as it has been an active model of research, containing a variety of features such as localization, resonance, dependence of the spectra on number theoretical properties, etc., and has been used as a model for a very wide range of physical systems. For a better understanding, we briefly review the quantum and classical mechanics of the kicked rotor in this section.

A. Kicked rotor: classical and quantum dynamics

The kicked rotor can be described as a pendulum subjected to periodic kicks (with period T) with the following Hamiltonian:

$$H = \frac{(p+\gamma)^2}{2} + K\cos(\theta + \theta_0) \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{4.1}$$

where K is the stochasticity parameter. The parameters γ and θ_0 are introduced in the Hamiltonian in order to mimic the effects of the time reversal (T) and the parity (P) symmetry breaking in the quantum system.

The related quantum dynamics can be described, by using Floquet's theorem, by a discrete time evolution operator $U = BG$, where $B = \exp(-iK\cos(\theta+\theta_0)/\hbar)$ and $G = \exp(-i(p+\gamma)^2/4\hbar)$. The nature of the quantum dynamics and therefore the statistical properties of the associated quantum operators depend on $\hbar$ and K. For a rational value of $\hbar T/2\pi$, the dynamics can be confined to a torus, while for irrational value it takes place on a cylinder. We employ torus boundary conditions ($q' = q + 2\pi$, $p' = p + 2\pi M/T$) by taking $\hbar T/2\pi = M/N$; both p and θ then have discrete eigenvalues, and U can be reduced to a finite N-dimensional matrix of the form [14–16]

$$U_{mn} = \frac{1}{N}\exp\left[-i\frac{K}{\hbar}\cos\left(\frac{2\pi m}{N} + \theta_0\right)\right] \times \sum_{l=-N_1}^{N_1} \exp[-i(\pi^2\hbar l^2 - \pi\gamma l)]\exp\left[-i\left(\frac{l(m-n)}{N}\right)\right], \tag{4.2}$$

where $N_1 = (N-1)/2$ (with N odd) and $m,n = -N_1, -N_1+1, \dots, N_1$.

The quantum dynamics has a time reversal symmetry T for $\gamma = 0$ and a parity symmetry P for $\theta_0 = 0$. Though the T symmetry may be violated for $\gamma \neq 0$, a still more generalized antiunitary symmetry $S = TP = PT$ can be preserved in the system if $\theta_0 = 0$ [14,2]. For $K^2 \gg N\hbar$, the quantum dynamics is delocalized in the phase space and two point spectral fluctuation measures have been shown to be well modeled by various symmetry classes of RMT [14,2]. In the opposite limit of weak chaos, namely, $K^2 \ll N\hbar$, the eigenstates localize in momentum space, and one obtains a Poisson distribution for the spectrum [2]. For numerical comparison of higher order measures with RMT, therefore, we choose various parameters such that condition $K^2 \gg N\hbar$ is always satisfied.

B. Numerical study of skewness and excess

In this section, we numerically study skewness and excess, that is, the third and fourth order spectral fluctuation measures, for the quantum kicked rotor (QKR) spectra and compare them with corresponding RMT results. Although the kth order form factor can easily be calculated for quantum maps [see Eq. (2.10)], we choose to study γ_1 and γ_2 as corresponding numerical results (for form factors) in RMT are not available. But, as both γ_1 and γ_2 can be analytically expressed in terms of third and fourth order correlations R_3 and R_4 (see Ref. [15] for these expressions) and, therefore, are related to form factors K_3 and K_4 [Eq. (2.8)], any conclusion about the validity of the RM model for the former measure will be applicable for the latter too.

Both skewness and excess are the functions of third and fourth order central moments of a distribution, respectively, which contain information about the probability of higher order events as compared to lower order. More precisely, skewness denotes the absence of symmetry in the distribution, and can be defined as follows:

$$\gamma_1(r) = \frac{\mu_3(r)}{\sigma^3(r)}. \tag{4.3}$$

The excess γ_2 describes the difference between the kurtosis values (i.e., the fourth central moment calculated in units of the square of the second central moment) of the distribution and that of a normal distribution (kurtosis is 3),

$$\gamma_2(r) = \frac{\mu_4(r)}{\sigma^4(r)} - 3, \tag{4.4}$$

where σ^2 is the variance in the number of levels in a length of r mean spacings, and μ_3 and μ_4 are corresponding third and fourth central moments. If the excess is less than zero, the curve is platykurtic, and, if it is positive, leptokurtic.

For both γ_1 and γ_2 studies, the spectral data consists of the eigenvalues of 50 matrices of dimension $N = 199$ obtained by diagonalizing the U matrix [Eq. (4.2)] for various values of K in the neighborhood of $K \simeq 20\,000$. The choice of such a high value of K is made to ensure the delocalization of quantum dynamics which makes U a full random matrix (see Ref. [2]). Due to strong sensitivity of the eigenvalues to small changes in K, these sequences of quasienergies can be regarded as mutually independent.

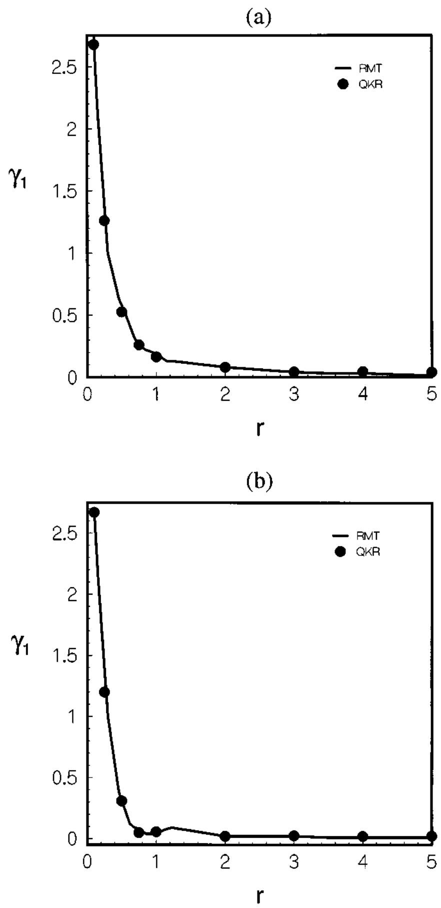

FIG. 1. The behavior of $\gamma_1(r)$ with respect to r, with $N=199$, $\hbar=1$, $T=1$, and $K=20\,002\rightarrow 20\,050$, and for (a) $\gamma=0.0$ and $\theta_0=\pi/2N$, TR preserved; (b) $\gamma=0.7071$ and $\theta_0=\pi/2N$, TR broken. The solid curve depicts the corresponding RMT behavior, namely, COE [in (a)] and CUE [in (b)] limits.

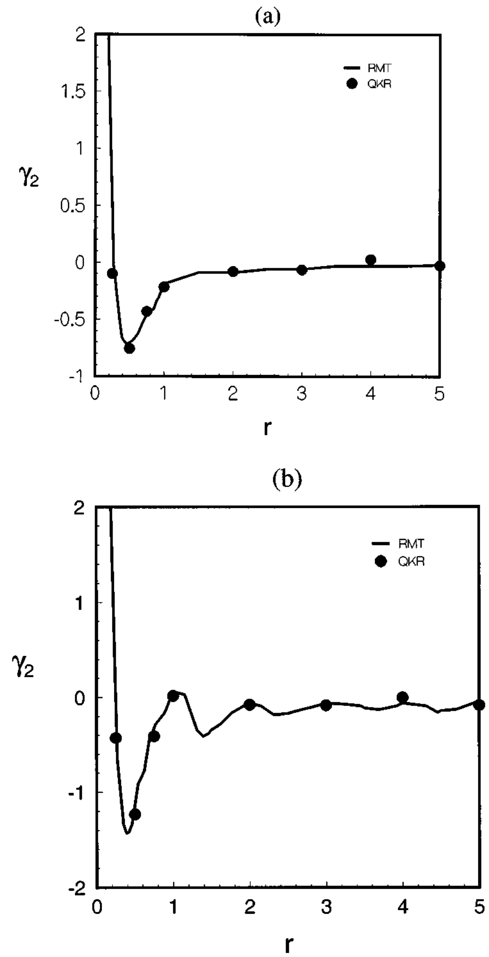

FIG. 2. The behavior of $\gamma_2(r)$ with respect to r, with $N=199$, $\hbar=1$, $T=1$, and $K=20\,002\rightarrow 20\,050$, and for (a) $\gamma=0.0$ and $\theta_0=\pi/2N$, TR preserved; (b) $\gamma=0.7071$ and $\theta_0=\pi/2N$, TR broken. The solid curve depict the corresponding RMT behavior, namely, COE [in (a)] and CUE [in (b)] limit.

The results obtained for γ_1 and γ_2 of the QKR spectra are displayed, as functions of r, in Figs. 1 and 2. For comparison, the corresponding RMT results (taken from Ref. [15]) are also given in each of the figures. The good agreement between QKR and RMT results indicated by each of these figures reconfirms our semiclassical results obtained in Sec. III. Furthermore, as indicated by these figures, the higher order correlations, e.g., third and fourth order, seem to be very weak, nearly zero, at very long energy ranges, and levels seem to be uncorrelated while strong correlation seem to be existing for short ranges. A continuously decreasing value of γ_1 implies the tendency of the distribution to appear more and more symmetric as the range of the distribution increases, and finally to acquire a Gaussian form for long energy ranges. This conclusion is also supported by our γ_2 study, which shows a larger probability of higher order events for QKR at small energy ranges (as compared to the Gaussian case) while, for large energy ranges, it acquires the same form for both.

V. CONCLUSION

We conclude this paper with a summary of our principal results, and a brief discussion of the open problems. We have shown that, in long time limits, the higher order spectral correlations in quantum chaotic maps can be well modeled by corresponding ones in RMT. By using the example of a kicked rotor, we have also verified this numerically. These results are also valid for conservative Hamiltonians; this follows due to similar expressions for the level density in terms of classical periodic orbits for both cases (see Ref. [4] for level density for autonomous case). The various summation formulas used to evaluate fluctuation measures such as sum over amplitudes (see the Appendix) whose derivation depends on the uniform distribution of periodic orbits and therefore only on the strongly chaotic nature of dynamics, still remain valid. Thus one obtains similar results for form factors in autonomous systems. This should not be surprising, as both Gaussian ensembles (the ensembles of Hermitian matrices and therefore of conservative Hamiltonians) and circular ensembles are known to have the same statistical behavior in semiclassical limit [16].

Although the study presented here deals with higher order correlations at a fixed value of the parameter, we also expect the validity of the random matrix model for higher order parametric correlations in quantum chaotic spectra, but again only on long time-scales. This intuition, based on the analogy of second order parametric density correlations for mesoscopic systems with a disordered potential, the one dimensional many body Hamiltonian (Calogero-Sutherland model), scattering systems, quantum chaotic systems, and random matrix models, as well as the existence of a common mathematical base (nonlinear σ model and supersymmetry approach) further encourages us to hope for the extension of this analogy to higher order correlations as well.

Ignorance about action correlations on very long time scales handicaps us from doing the same analysis on these scales. It will also be of interest to study these correlations on short time scales. It is on these scales where second order spectral correlations deviate from RMT and show a nonuniversal behavior. One expects to see similar deviations for higher orders too. For a complete understanding of higher order correlations in quantum chaotic systems, therefore, one should study the action and periodic orbit correlations; this will give us some insight into the behavior of fluctuations in other systems (above mentioned) too. We intend to do so in the future.

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APPENDIX: CALCULATION OF $F(A,B)$

To evaluate the sum

$$f(a,b)=\sum_j \frac{A_j^a}{n_j^b}\,\delta\left(|\tau|-\frac{n_j}{N}\right), \tag{A1}$$

where A_j is the amplitude and n_j is the period of the orbit in classical phase space, we remind ourselves that, in phase space, the orbits proliferate exponentially with time and the density of distribution of periods over long orbits is given by $\exp(2\gamma|n|)/|n|$, with γ as the entropy of the classical motion in large time limits. This, along with the approximation $A_j \simeq n_j \exp(-\gamma|n_j|)$, enables us to make the following replacement:

$$\sum_j \frac{A_j^a}{n_j^b}\,\delta\left(|\tau|-\frac{n_j}{N}\right) \simeq \sum_{n=0}^{\infty} \exp(-\gamma n(a-2))n^{a-b-1} \times \delta\left(|\tau|-\frac{n}{N}\right), \tag{A2}$$

which gives

$$f(a,b)=\exp(-\gamma N|\tau|(a-1))|\tau|^{a-b-1}N^{a-b}. \tag{A3}$$

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